Bushby Manuscript

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Contents

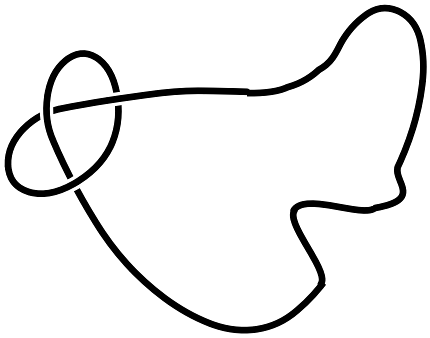
Chapter 1

Introduction to Knot Theory

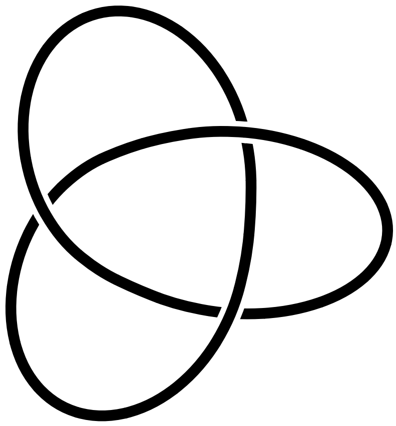
Before we look to Bushby's Manuscript I would like to define a few of the terms that will be used later on and give a brief introduction to Knot Theory.

Firstly, we will start by defining what a knot is. In Knot Theory, we consider **knots** as closed loops. If we tied a knot in a piece of string we would then glue the ends together so that the knot cannot be undone by pushing the end back through the knot. In this way, we can study knots and classify them.
A **link** is a knot which is made up of two different closed loops linked together.
We usually describe knots (and links) by way of a **knot diagram**. This is a two-dimensional drawing of the knot in which we can see the over and under crossings of the knot.

In these diagrams we can shrink a strand of a knot or smooth it out, performing what are called **planar isotopies**. In this way we are not allowed to undo crossings.



A trefoil before planar isotopy.

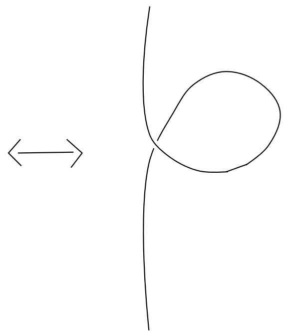


A trefoil after we have smoothed out and shrunk a strand.

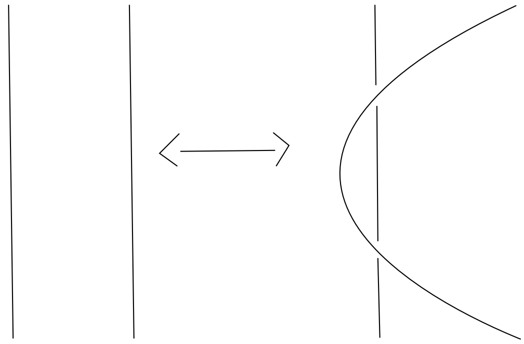
What we would like to do is classify the possible knots, but we have a difficulty in determining when two knots are the same. We need a tool by which to identify when two knots are the same.

Reidemeister's Theorem [1]: Two knot diagrams belong to the same knot if they can be related by a sequence of Reidemeister moves and planar isotopies. There are three Reidemeister moves which are described as follows;

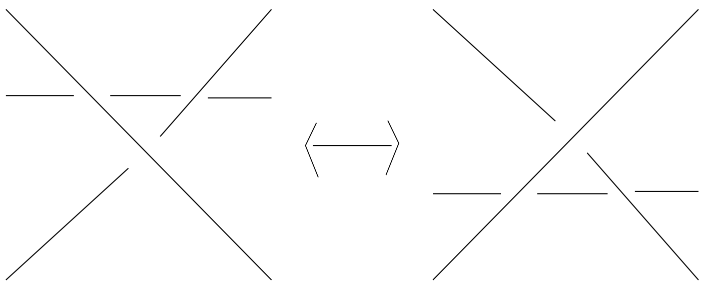
Type I - Add or remove a twist in either direction.



Type II - Move one strand completely over or under another.



Type III - Move a strand completely over or under a crossing.



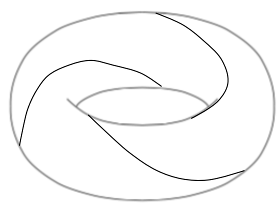
These moves act on a small region of the knot diagram and leave the rest unchanged.

For a knot, we describe it based on the minimum number of crossings for any projection of the knot. This is usually found by acting on a knot diagram by Reidemeister moves to reduce the number of crossings.

Knots can be one of two types, **alternating** or **non-alternating**. To determine if your knot is alternating take the projection of the knot with the minimum number of crossings and follow a strand of the knot around the knot from start to finish. If the strand goes over and under the crossings in an alternating fashion then the knot is alternating, otherwise it is non-alternating.

Another definition we will need later on is the definition of a Torus Knot.
A (p, q)**-Torus Knot** is obtained by looping a string through the hole of a torus p times with q revolutions before joining its ends, where p and q are relatively prime, meaning their only common divisor is 1. A property of a Torus Knot is that a (p, q)-Torus Knot is equivalent to a (q, p)-Torus Knot.
If p and q are not relatively prime we get a Torus Link, with the number of the components of the link being equal to $gcd(p, q)$.
As we have a choice of direction for the revolutions q we can sometimes end up with the mirror image of the knot (p, q) which we denote $(p, -q)$ when the knot's mirror images are distinct. This means they cannot be deformed between each other using Reidemeister moves.

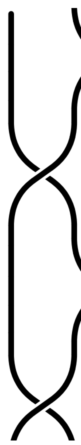
What does a Torus Knot look like?



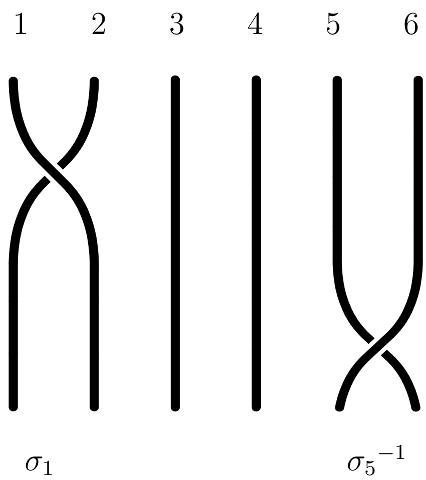
This is the $(3, 2)$-Torus Knot. We can see the string has been looped through the hole of the torus 3 times, the string is taken around the torus twice. This shows the top view of the Torus Knot, when we take the string off the torus we will get the Trefoil Knot.

Another definition we will need later is that of a Braid.
A **Braid** is formed by taking a number of pieces of string and passing them over

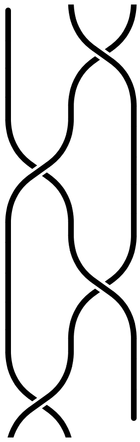
each other, in a similar method to a plait. The strings are thought of as being attached to a horizontal bar at the top, so that they cannot be moved, and when we are passing the strings over each other, we are not allowed to double any string back on itself.



We number the strands of the braid $(1, \cdots, n)$ and describe the passing of the strands over each other as follows. When we pass strand n over strand $n+1$, we call that a σ_n move. If strand $n+1$ is passed over strand n, we call that a ${\sigma_n}^{-1}$ move.

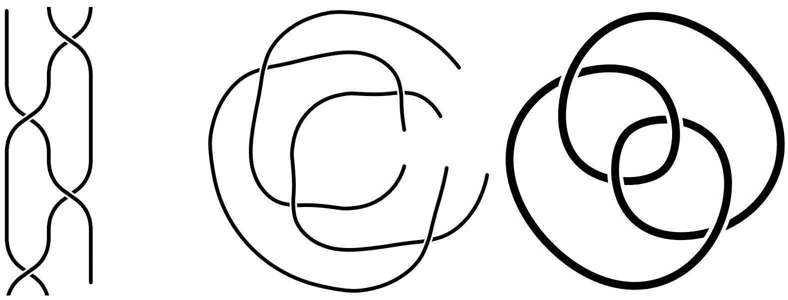


In this way, by labelling each crossing of a braid in turn, we end up with a word that describes that braid.



This braid can be described by the word $\sigma_2{\sigma_1}^{-1}\sigma_2{\sigma_1}^{-1}$, which can also be written as $(\sigma_2{\sigma_1}^{-1})^2$.

We call the **closure of the braid** the knot resulting from joining the top of the strand in a given position to the bottom of the strand in the same position.



By closing the braid $\sigma_2{\sigma_1}^{-1}\sigma_2{\sigma_1}^{-1}$ we see that we end up with the Figure Eight Knot.

Now we can move to examining the Knot Theory in Bushby's Manuscript.

Chapter 2

Volume 2

2.1 Bushby's Plait Knots

In Volume 2 Bushby begins to discuss **Plait Knots**. He describes the drawing of them and gives them a notation of the form $S^n xb/yc$ where n relates to the series of the knot and determines the number of concentric circles to be drawn, x is the number of bights and y the number of crossings.
This description bears a similarity to the definition of a Torus Knot.
Looking at Bushby's definition we see a similarity between the number of concentric circles and the number of revolutions q described in the Torus Knot definition.

Let's now look at Bushby's Plait Knots in turn. I will list his notation for the knot and what it relates to mathematically.

Series 1

$S^1 1b/0c$ - This is the unknot or trivial knot, also known as the 1-sphere S^1 or the circle.

Series 2

$S^2 1b/1c$ - This again is the unknot with a Type 1 Reidemeister move acting on it, forming a twist. We can see that this twist could easily be twisted out resulting in the unknot.

$S^2 2b/2c$ - This is called the Hopf Link and is Torus Link $(2, 2)$.

$S^2 3b/3c$ - This is the Trefoil, Torus Knot $(3, 2)$.

$S^2 4b/4c$ - This is Torus Link $(4, 2)$.

$S^2 5b/5c$ - This is the Cinquefoil, Torus Knot $(5, 2)$.

$S^2 6b/6c$ - This is Torus Link $(6, 2)$.

$S^2 7b/7c$ - This is the Septafoil, Torus Knot $(7, 2)$.

$S^2 8b/8c$ - This is the Torus Link $(8, 2)$.

$S^2 9b/9c$ - This is the Nonafoil, Torus Knot $(9, 2)$.

$S^2 10b/10c$ - This is Torus Link $(10, 2)$.

$S^2 11b/11c$ - This is Torus Knot $(11, 2)$.

$S^2 12b/12c$ - This is Torus Link $(12, 2)$.

We can see for most of these knots and links that Bushby's notation of $S^n xb/yc$ corresponds to a Torus Knot or Link with notation (x, n). In this way Bushby seems to have listed the first 12 Knots and Links which traverse the Torus twice.

Let's now look to Bushby's knots with series S^3.

Series 3

$S^3 2b/4c$ - This is the Figure Eight Knot but is not a Torus Knot. However it is quite common for people to mistakenly refer to the Figure Eight Knot as a Torus Knot.

$S^3 1b/2c$ - This again is the unknot, now with two Type 1 Reidemeister moves applied to it.

$S^3 3b/6c$ - This is the Borromean Link or Borromean Rings, again not a Torus Knot but looks similar to Torus Knot $(3, 3)$.

$S^3 4b/4c$ - This is knot 8.18 on the Knot Atlas knot tables [2], again not a Torus Knot but looks similar to Torus Knot $(4, 3)$.

$S^3 5b/10c$ - This is knot 10.123 on the Knot Atlas knot tables, again not a Torus Knot but looks similar to Torus Knot $(5, 3)$.

The knots and links from here onwards are too big too be included in the knot tables and given a formal name. None of the following knots or links are Torus so cannot be classified in the notation above and here is why.

In Bushby's description on how to draw the Plait Knots he specifies they must be drawn with the strands going over and under alternately, hence forming alternating knots. When it comes to Torus Knots, only Torus Knots where $q = 2$ are alternating. All others have at least one crossing which prevents the knot being alternating. [3]
In this way Bushby has described knots which look like Torus Knots but are alternating.

2.2 Properties of knots

Under his drawings of these knots, Bushby begins to discuss a few properties of knots. He begins by stating the first knot of any series is a Twist Knot and can be undone or reduced to the unknot by one twist. This is similar to the fact that a Torus Knot is trivial or the unknot if and only if p or q are equal to 1 or -1.

He also correctly states that there are infinitely many knots then, incorrectly (but then crosses out), states that every knot can be reduced to an alternating knot. If you'd like to convince yourself that this is untrue, try reducing a non-alternating knot to an alternating one using Reidemeister moves.

He describes the Plait Knots discussed above as **Regular** and states that reduced knots are either Regular or Irregular, where Regular means they can be represented on cords laid around the centre of a circle and Irregular if they cannot be drawn in this way. This definition of a Regular Knot sounds similar to the definition of a Torus Knot and it is true that a knot is either Torus or it is not, a non-Torus Knot cannot be deformed by Reidemeister moves into a Torus Knot.

Bushby then discusses the crossing number of his Regular Knots and tries to link this to the series number and the number of bights. His formula for knots $S^n xB/yC$ states that $y = x * (n-1)$ which he has not managed to prove.
This statement is similar to saying that the crossing number of a Torus Knot is equal to $p*(q-1)$ or $q*(p-1)$, as a (p,q)-Torus Knot is the same as a (q,p)-Torus Knot.
In fact, the crossing number of a (p,q)-Torus Knot (or $(p,-q)$-Torus Knot) is equal to $min((p-1)*|q|, (|q|-1)*p)$ which isn't far off what Bushby had! This was proven by Murasagi in 1991. [4]

Next Bushby looks to try to determine the possible Regular Knots for a given crossing number. He starts by taking a crossing number and then finding its divisors, then putting them into his formula relating bights, crossings and the series number in order to determine which Plait Knots are possible.
The same procedure can be used for Torus Knots if we remember the formula for crossing number from earlier.

Bushby then looks to the number of cords a given Regular Knot is on. He states that the number of cords is equal to the Greatest Common Divisor of the series number and number of bights. This fact is true for Torus Knots as well!
He then looks to determine the number of Regular Knots on a certain number of cords with a given number of crossings which then completes his study of Regular Knots.

Chapter 3

Volume 4

In Volume 4 Bushby begins some **Experiments in Twisting**. He talks of arranging a number of cords, equidistant from each other, on the circumference of a circular tube, then twisting and joining the ends to produce a knot. The way in which the cords are twisted can be thought of as the strands of a braid crossing over each other.

If we consider this tubular twisting as a braid, we can rewrite each "arc", described by Bushby as a $360/z^\circ$ twist where z is the number of cords on the tube, as a braid word.
A revolution through one "arc", can be written as braid word $\sigma_1\sigma_2\cdots\sigma_{z-2}\sigma_{z-1}$.
A revolution through n "arcs", can be written as braid word $(\sigma_1\sigma_2\cdots\sigma_{z-2}\sigma_{z-1})^n$.

Closing these braids will result in a knot or a link, and so we can analyse Bushby's discussions of his "Experiments in Twisting".

Two cords
One arc - Braid σ_1. Bushby gives this as $S^2 1b/1c$, which we know from the previous section, is the unknot.
Two arcs - Braid $(\sigma_1)^2$. Given as $S^2 2b/2c$, which is the Hopf Link and Torus Link $(2, 2)$.
Three arcs - Braid $(\sigma_1)^3$. Given as $S^2 3b/3c$, which is the Trefoil, Torus Knot $(3, 2)$.
Four arcs - Braid $(\sigma_1)^4$. Given as $S^2 4b/4c$, which is Torus Link $(4, 2)$.

Five arcs - Braid $(\sigma_1)^5$. Given as $S^2 5b/5c$, which is the Cinquefoil, Torus Knot $(5, 2)$.
Six arcs - Braid $(\sigma_1)^6$. Not given, but corresponds to $S^2 6b/6c$ which is Torus Link $(6, 2)$.

Three cords

One arc - Braid $\sigma_1\sigma_2$. Given as $S^3 1b/2c$, which is the unknot. Two arcs - Braid $(\sigma_1\sigma_2)^2$. Given as $S^2 3b/3c$, which is the Trefoil, Torus Knot $(3, 2)$.
Three arcs - Braid $(\sigma_1\sigma_2)^3$. Called $K^3 3a/6c$ to denote three arcs and six crossings, but is Torus Link $(3, 3)$.
Four arcs - Braid $(\sigma_1\sigma_2)^4$. Called $K^3 4a/8c$, and is Torus Knot $(4, 3)$.
Five arcs - Braid $(\sigma_1\sigma_2)^5$. Called $K^3 5a/10c$, and is Torus Knot $(5, 3)$.
Six arcs - Braid $(\sigma_1\sigma_2)^6$. Called $K^3 6a/12c$, and is Torus Knot $(6, 3)$.

From these results we see that Bushby's experiments are building Torus knots and links. Why do these braids correspond to Torus knots and links?
Any (p, q) Torus knot or link can be made from a closed braid with p strands, with braid word $(\sigma_1\sigma_2 \cdots \sigma_{p-2}\sigma_{p-1})^q$ [5], which is exactly what we have.

In Volume 2 we saw Bushby almost construct the Torus knots and links and prove some key properties about them. In Volume 4 we see he has come up with an algorithm to construct them. The knots in both volumes look very similar, but most just differ by the fact that most in Volume 2 are alternating whereas most in Volume 4 are not.

From Bushby's work we see he has put considerable effort into categorising knots, even coming across an algorithm for generating Torus knots and links and creating a categorisation for them.

Bibliography

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[3] T. Abe, “An estimation of the alternation number of a torus knot,” *Journal of Knot Theory and its Ramifications*, vol. 18, no. 3, pp. 363–379, 2009.

[4] K. Murasugi, “On the braid index of alternating links,” *Trans. Amer. Math. Soc.*, vol. 326, no. 1, pp. 237–260, 1991.

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